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Poisson–Lie structures on Poincaré and Euclidean groups in three dimensions

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Abstract. The complete list of Poisson–Lie structures on Poincaré and Euclidean groups in three dimensions is presented. Some new solutions for inhomogenous $SO(p,q)$ are given.

1. Introduction

The problem of classification of Poisson–Lie structures on inhomogenous $SO(p,q)$ groups (i.e. the semidirect product of $SO(p,q)$ and R^{p+q}, where the action is given by fundamental representation) is not yet solved in general. It was investigated in [1] where it was shown that all Poisson–Lie structures on these groups (for $p+q>2$) are coboundary. Also a number of solutions were presented, especially for inhomogenous $SO(1,3)$ but even for this group it is not known whether other solutions exist. If we think of Poisson–Lie groups as 'classical limits' of quantum groups, it is reasonable, as a first step to find quantum deformations, to study the possible Poisson–Lie structures on a given group and then select those which can be 'quantized' on the C^*-algebra level [4]. In this paper we give the complete list of bialgebra structures on inhomogenous $so(3)$ and $so(1,2)$ algebras. The structures were found 'by force' so the paper is rather technical. Nevertheless we were able to find a new solution for any dimension. These solutions come from the general construction which will be described elsewhere [5] and are presented in the remarks within section 3.

2. Basic definitions and notation

Let (V,η) be a three-dimensional, real vector space with symmetric, nondegenerate, bilinear form η. In fact, we have two situations: either the signature of η is (3, 0)—this corresponds to the Euclidean group or the signature is (1, 2)—this corresponds to the Poincaré Group. By η we also denote isomorphism $V\simeq V^*$ given by: $\eta_x(y):=\eta(x,y)$. A^t is the transposition defined by $\eta:\ \eta(Ax,y)=:\eta(x,A^ty)$. Let $H:=SO(\eta)$ be the special orthogonal group and $\mathfrak{h}:=so(\eta)$ its Lie algebra. H acts on V by fundamental representation. Let G be a semidirect product defined by this action and $\mathfrak{g}$ its Lie algebra.

We use the following natural isomorphisms intertwining with the corresponding representations of H. Ω: $V\wedge V\to\mathfrak{h}$: $\Omega_{x,y}:=\Omega(x\wedge y):=x\otimes\eta(y)-y\otimes\eta(x)$. For any orthonormal basis (e_1,e_2,e_3) we denote: $k_1:=\Omega_{e_2,e_3}$, $k_2:=\Omega_{e_3,e_1}$, $k_3:=\Omega_{e_1,e_2}$.

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Note that Ω viewed as an element of $(\bigwedge^2 V)^* \otimes \mathfrak{h} \simeq \bigwedge^2 V \otimes \mathfrak{h} \subset \bigwedge^3 \mathfrak{g}$ is G-invariant [1]. In any basis:

$$\Omega = \eta^{ij}\eta^{kl} e_j \wedge e_k \otimes \Omega_{jl}$$
$$\pi : V \wedge V \to V : \eta(\pi(x \wedge y))(z) := \mathrm{Vol}(x \wedge y \wedge z)$$

where Vol is volume form on V determined (up to a sign) by η. In any orthonormal, oriented basis: $\pi(e_i \wedge e_j) = \Sigma_k \epsilon_{ijk}\eta_{ii}\eta_{jj} e_k$.

These two isomorphisms provide us with further identifications:

$$V \otimes \mathfrak{h} \simeq V \otimes V \simeq V \otimes V^* \simeq \mathrm{End}(V), e_i \otimes k_j \mapsto \eta_{11}\eta_{22}\eta_{33} e_i \otimes e^j.$$

$$\bigwedge^2 V \otimes \mathfrak{h} \simeq V \otimes \mathfrak{h} \simeq \mathrm{End}(V), e_i \wedge e_j \otimes k_l \mapsto \Sigma_k \epsilon_{ijk}\eta_{kk} e_k \otimes e^l.$$

$$V \otimes \bigwedge^2 \mathfrak{h} \simeq V \otimes \bigwedge^2 V \simeq V \otimes \mathfrak{h} \simeq \mathrm{End}(V), e_i \otimes k_j \wedge k_l \mapsto \Sigma_k \epsilon_{jlk}\eta_{kk} e_i \otimes e^k.$$

For any orthonormal basis (e_1, e_2, e_3) we denote: $e_- := \frac{1}{\sqrt{2}}(e_1 - e_2)$, $e_+ := \frac{1}{\sqrt{2}}(e_1 + e_2)$.

H acts on $\mathrm{End}(V)$ by conjugation and results in the decomposition into irreducible subspaces:

$$\mathrm{End}(V) = W_0 \oplus W_1 \oplus W_2 \tag{1}$$

where: $W_0 := \{\lambda \mathrm{id}_V : \lambda \in R\}$, $W_1 := \mathfrak{h}$, $W_2 := \{a \in \mathrm{End}(V) : a^t = a \text{ and } \mathrm{Tr}(a) = 0\}$.

It is known that Poisson–Lie structures on G are coboundary [1], i.e. are given by some element $r \in \bigwedge^2 \mathfrak{g}$ satisfying equation $[r, r] \in (\bigwedge^3 \mathfrak{g})_{inv}$ (where $[r, r]$ is the Schouten bracket).

Note the following lemma.

Lemma 1. *$(\bigwedge^3 \mathfrak{g})_{inv}$ is two-dimensional and is spanned by elements: $\Omega \in \bigwedge^2 V \otimes \mathfrak{h}$ and $\tilde{\eta} := e_1 \wedge e_2 \wedge e_3 \in \bigwedge^3 V$.*

Proof.

$$\bigwedge^3 \mathfrak{g} = \left(\bigwedge^3 V\right) \oplus \left(\bigwedge^2 V \otimes \mathfrak{h}\right) \oplus \left(V \otimes \bigwedge^2 \mathfrak{h}\right) \oplus \left(\bigwedge^3 \mathfrak{h}\right). \tag{2}$$

This decomposition is H-invariant and we have the following H-invariant subspaces:

$\bigwedge^3 V$—this is one dimensional H-invariant subspace;

$\bigwedge^2 V \otimes \mathfrak{h} \simeq \mathrm{End}(V) = W_0 \oplus W_1 \oplus W_2$—so the only H-invariants elements are in the image of W_0 and this is subspace spanned by Ω;

$$V \otimes \bigwedge^2 \mathfrak{h} \simeq \mathrm{End}(V) = W_0 \oplus W_1 \oplus W_2$$

so as above the only H-invariants elements are in the image of W_0.

$\bigwedge^3 \mathfrak{h}$—this is a one-dimensional H-invariant subspace

One can check that only the first two subspaces are V-invariant. □

Let $r = a + b + c$ be a decomposition corresponding to:

$$\bigwedge^2 \mathfrak{g} = \left(\bigwedge^2 V\right) \oplus \left(V \otimes \mathfrak{h}\right) \oplus \left(\bigwedge^2 \mathfrak{h}\right) \tag{3}$$

then $[r, r] = 2[a, b] + (2[a, c] + [b, b]) + 2[b, c] + [c, c]$ is the decomposition corresponding to (2). From the lemma above it follows that we have to solve equations:

$$[a, b] = p\tilde{\eta} \qquad 2[a, c] + [b, b] = \mu\Omega \qquad p, \mu \in R \tag{4}$$
$$[b, c] = 0 \qquad [c, c] = 0. \tag{5}$$

3. Results

The main results of this paper are shown in the following complete list of solutions (up to automorphisms of $\mathfrak{g}$) of equations (4) and (5) for $P(3)$ and $E(3)$.

3.1. Poisson structures on P(3)

(I) $c = \frac{1}{\sqrt{2}}(k_1 + k_2) \wedge k_3$, $b = \alpha(e_1 \wedge k_1 + e_2 \wedge k_2 + e_3 \wedge k_3)$, $a = 0$
$\alpha = 0, 1,\ \mu = 2\alpha^2,\ p = 0$.
In the remaining cases $c = 0$.
(IIa) $b = \rho e_3 \wedge k_3 + \alpha(e_2 \wedge k_1 + e_1 \wedge k_2)$, $a \in \bigwedge^2 V$
$\rho \geqslant 0$, $\alpha = 0, 1$, $\alpha^2 + \rho^2 \neq 0$, $\mu = -2\alpha^2,\ p \in R$.
(IIb) $b = \rho e_1 \wedge k_1 + \alpha(e_3 \wedge k_2 - e_2 \wedge k_3)$, $a \in \bigwedge^2 V$
$\rho \geqslant 0$, $\alpha = 0, 1$, $\alpha^2 + \rho^2 \neq 0$, $\mu = 2\alpha^2,\ p \in R$.
(IIc) $b = \alpha \frac{1}{\sqrt{2}}(e_3 \wedge (k_1 + k_2) + (e_1 - e_2) \wedge k_3) + \rho(e_1 - e_2) \wedge (k_1 + k_2)$, $a \in \bigwedge^2 V$
$\alpha = 0, 1$, $\rho \geqslant 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = 0,\ p \in R$.
(IIIa) $b = \frac{1}{\sqrt{2}}(e_1 - e_2) \wedge k_3$, $a \in \bigwedge^2 V$
$\mu = 0,\ p \in R$.
(IIIb) $b = e_1 \wedge k_1 + (\rho - 1)e_2 \wedge k_1 + (\rho + 1)e_1 \wedge k_2 - e_2 \wedge k_2 + \rho e_3 \wedge k_3$, $a \in \bigwedge^2 V$
$\rho \in R \setminus \{0\}$, $\mu = 2\rho^2,\ p \in R$.
(IV) $b = e_1 \wedge k_1 + e_2 \wedge k_2 + e_3 \wedge k_3$, $a = 0$
$\mu = 2,\ p = 0$.
(V) $b = 0$, $a \in \bigwedge^2 V$
$\mu = 0,\ p = 0$.

3.2. Poisson structures on E(3)

In all cases $c = 0$.
(I) $b = \alpha(e_1 \wedge k_2 - e_2 \wedge k_1) + \rho e_3 \wedge k_3$, $a \in \bigwedge^2 V \alpha = 0, 1$, $\rho \geqslant 0$, $\alpha^2 + \rho^2 \neq 0$, $\mu = -2\alpha^2,\ p \in R$.
(II) $b = e_1 \wedge k_1 + e_2 \wedge k_2 + e_3 \wedge k_3$, $a = 0$. $\mu = 2,\ p = 0$.
(III) $b = 0$, $a \in \bigwedge^2 V$. $\mu = 0,\ p = 0$.

We can still use automorphisms of $\mathfrak{g}$ generated by some vectors from V to restrict the possible forms of a.

For $P(3)$:

(IIa) Using a two-parameter group of automorphisms of $\mathfrak{g}$ generated by e_1 and e_2 we can transform this solution to solutions with: $a = a_3 e_1 \wedge e_2$ for $\alpha \neq \rho$ or $a = a_3 e_1 \wedge e_2 + a_- e_3 \wedge e_-$ for $\alpha = \rho$. After such transformation $p = -2a_3\alpha$, $N = 2$ (N denotes the number of parameters in the solution).

(IIb) Here we use e_2 and e_3 and obtain $a = a_1 e_2 \wedge e_3$. In this case $p = -2a_1\alpha$, $N = 2$.

(IIc) Now using e_+ and e_3 we obtain $a = a_+ e_3 \wedge e_+$; $p = 2a_+$, $N = 2$.

(IIIa) Using e_- and e_+ we obtain $a = a_+ e_3 \wedge e_+ + a_- e_3 \wedge e_-$, $p = a_+$, $N = 2$.

(IIIb) As above, using e_- and e_+ we obtain $a = a_3 e_1 \wedge e_2 + a_+ e_3 \wedge e_+$, $p = -2\rho a_3$, $N = 3$.

(V) Using isomorphism $\bigwedge^2 V \simeq V$ and dilations we can assume that a is of one of the following forms: $e_2 \wedge e_3$, $e_1 \wedge e_2$, $e_3 \wedge e_-$, $N = 0$.

For E(3):

(I) Using e_1 and e_2 we can always put: $a = a_3 e_1 \wedge e_2$, $p = -2\alpha a_3$, $N = 2$.

(III) Using isomorphism $\bigwedge^2 V \simeq V$ and dilations we can assume that $a = e_1 \wedge e_2$, $N = 0$.

Remark 1. Above we indicate the value of p since $p = 0$ is a necessary and sufficient condition for the existence of Poisson Minkowski space. [3]

Remark 2. Solutions (IV) for $P(3)$ and (II) for $E(3)$ are directly connected to the dimension three and have no counterparts in higher dimension.

Remark 3. Solutions (II) for $P(3)$ and (I) for $E(3)$ are of the well known form [1] valid for arbitrary inhomogenous $so(p, q)$. Namely, for each $z \in V$ let $b := b_z := \eta^{jk} e_j \otimes \Omega_{z,e_k}$. Then $[b, b] = -\eta(z, z)\Omega$ and these are solutions (IIa–c) for $\alpha = 0$ and where z is respectively positive, negative and null vector.

Also if $b := b_z + z \wedge Z$ where $Z \in \mathfrak{h}$ such that $Zz = 0$. Then $[z \wedge Z, z \wedge Z] = [b_z, z \wedge Z] = 0$ and again $[b, b] = -\eta(z, z)\Omega$. Solutions (IIa–c) for $\alpha = 1$ correspond respectively to $z := e_3$, $Z := \rho k_3$, $z := e_1$, $Z := \rho k_1$, $z := e_-$, $Z := \sqrt{2}\rho(k_1 + k_2)$ and solution (I) for $E(3)$ corresponds to $z := e_3$, $Z := \rho k_3$.

Note also that these solutions (with $\rho = 0$) are tangent lifts of Poisson structure [2] on $SO(1, 2)$ and $SO(3)$ if we identify $P(3)$ and $E(3)$ with tangent groups: $TSO(1, 2)$ and $TSO(3)$.

Remark 4. Solution (III) for $P(3)$ can also be written in a form which gives us new solutions for $so(p, q)$.

(IIIa) Let $b := b_z + z \wedge Z + v \wedge Z$ where v is such that: $Zv = -z$ (it follows that z must be a null vector). We compute the brackets: $[v \wedge Z, v \wedge Z] = -2v \wedge z \wedge Z$, $[v \wedge Z, z \wedge Z] = 0$, $[v \wedge Z, b_z] = v \wedge z \wedge Z$. So $[b, b] = [b_z + z \wedge Z, b_z + z \wedge Z] = 0$. Solution (IIIa) corresponds to: $z := e_-$, $Z := \frac{1}{\sqrt{2}}(k_1 + k_2)$, $v := -(e_- + e_3)$.

More generally: let $b := b_z + \sum_i (z + v_i) \wedge Z_i$, where $Z_i z = 0$, $Z_i v_j = -\delta_{ij} z$, $[Z_i, Z_j] = 0$. Then $[b, b] = [b_z, b_z]$. For example if $\mathfrak{h} = so(1, n)$ one can take: $z := e_1 - e_{n+1}$, $v_i := e_i$, $Z_i := \Omega_{1,i} + \Omega_{i,n+1}$, $i = 2, \ldots, n$.

(IIIb) Let b be as above, but now we choose v such that: $Zv = v$. Then $[b, b] = [b_z + z \wedge Z, b_z + z \wedge Z] + 2[v \wedge Z, b_z + z \wedge Z] + [v \wedge Z, v \wedge Z]$. Now $[v \wedge Z, v \wedge Z] = 0$ and $[v \wedge Z, b_z] = v \wedge Z(b_z) - Z \wedge v(b_z) = v \wedge b_{Zz} - Z \wedge (-v \wedge z) = v \wedge z \wedge Z$, $[v \wedge Z, z \wedge Z] = -Z \wedge Zv \wedge z = -v \wedge z \wedge Z$. So we have $[b, b] = [b_z + z \wedge Z, b_z + z \wedge Z] = -\eta(z, z)\Omega$. These are solutions (IIIb) for $\rho > 0$ (section 5.2.1).

4. Computation of the Schouten bracket $[r, r]$.

To solve equations (4) and (5) we compute the bracket $[r, r]$ explicitly using identifications from section 1 and the fact that the bracket intertwines the corresponding representations of $\mathfrak{h}$. We obtain a system of equations on $\mathrm{End}(V)$.

Let us define mappings $\mathrm{End}(V) \otimes \mathrm{End}(V) \longrightarrow \mathrm{End}(V)$:

$F_0(a \otimes b) := \mathrm{Tr}(a^t b)\mathrm{id}_V$, then $F_0(a \otimes b) = F_0(b \otimes a)$ and F_0 intertwines representation on $\mathrm{End}(V) \otimes \mathrm{End}(V)$ with trivial representation on W_0.

$F_1(a \otimes b) := a^t b - b^t a$, then $F_1(a \otimes b) = -F_1(b \otimes a)$ and F_1 intertwines representation on $\mathrm{End}(V) \otimes \mathrm{End}(V)$ with representation on W_1.

$F_2(a \otimes b) := a^t b + b^t a - \frac{2}{3}\mathrm{Tr}(a^t b)\mathrm{id}_V$ then $F_2(a \otimes b) = F_2(b \otimes a)$ and F_2 intertwines representation on $\mathrm{End}(V) \otimes \mathrm{End}(V)$ with representation on W_2.

$r = a + b + c \in (\bigwedge^2 V) \oplus (V \otimes \mathfrak{h}) \oplus (\bigwedge^2 \mathfrak{h}) \simeq W_1 \oplus \mathrm{End}(V) \oplus W_1$

Let $b = x + y + t \in W_2 \oplus W_1 \oplus W_0$ be the decomposition (1).

Then $[r, r] = 2[b, a] + 2[c, a] + [b, b] + 2[b, c] + [c, c] = 2([x, a] + [y, a] + [t, a]) + 2[c, a] + ([x, x] + [y, y] + [t, t] + 2[x, y] + 2[x, t] + 2[y, t]) + 2([x, c] + [y, c] + [t, c]) + [c, c]$.

Next, each term is computed separately and brackets are expressed as combinations of F_0, F_1, F_2. The detailed computations are given in the appendix. It results that the equations (4) and (5) are equivalent to the following system of equations on $\mathrm{End}(V)$.

$$\mathrm{Tr}(C^2) = 0 \tag{6}$$
$$\mathrm{Tr}(XC) = 0 \tag{7}$$
$$XC - CX + 3(YC + CY) = 0 \tag{8}$$
$$(-XC - CX) + (YC - CY) = 0 \tag{9}$$
$$\tfrac{2}{3}\,\mathrm{Tr}(CA) + \mathrm{Tr}(X^2) - \tfrac{1}{3}\,\mathrm{Tr}(Y^2) + 2t^2 = \mu \tag{10}$$
$$-(CA - AC) + 2(XY + YX) + 4tX = 0 \tag{11}$$
$$CA + AC - \tfrac{2}{3}\,\mathrm{Tr}(CA)\mathrm{id}_V - 4(Y^2 - \tfrac{1}{3}\,\mathrm{Tr}(Y^2)\mathrm{id}_V) - 2(YX - XY) + 4tY = 0 \tag{12}$$
$$\mathrm{Tr}(XA) = p \tag{13}$$

$A^t = -A$, $C^t = -C$, $X^t = -X$, $Y^t = Y$, $\mathrm{Tr}(Y) = 0$, $T =: t\mathrm{id}_V$.
Capital letters A, C, X, Y, T denote elements of $\mathrm{End}(V)$ corresponding to the terms denoted by small letters in decomposition of r.

5. Solutions for $P(3)$

5.1. Solutions for $C \neq 0$

Equation (6) means that C is antisymmetric with null kernel so one can choose a basis (e_-, e_+, e_3) such that $C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. C is invariant under a one-parameter subgroup of $SO(1,2)$ stabilizing e_-. On the chosen basis this group acts as follows:

$$\begin{cases} e_- \mapsto e_- \\ e_+ \mapsto \dfrac{r^2}{2} e_- + e_+ + re_3 \qquad r \in R \; \cdot \\ e_3 \mapsto re_- + e_3 \end{cases} \tag{14}$$

Going back to $\bigwedge^2 \mathfrak{h}$ we have $c = \frac{1}{\sqrt{2}}(k_1 + k_2) \wedge k_3 = \Omega_{e_3,e_-} \wedge \Omega_{e_-,e_+}$.

Before we move on to the next equations, let us note that if $v \in V$ then the action of automorphism generated by v on c is given by: $v(c) = c - \frac{1}{\sqrt{2}} k_3 v \wedge (k_1 + k_2) + \frac{1}{\sqrt{2}}(k_1 v + k_2 v) \wedge k_3 + \frac{1}{\sqrt{2}}(k_1 v + k_2 v) \wedge k_3 v$. Using appriopriate v we can always assume that $X \neq 0$ and b contains no terms $e_1 \wedge k_2$ and $e_2 \wedge k_1$.

From equations (8) and (9) it follows that $\ker C$ is invariant under X and Y. From equation (7) it follows that $\ker X$ is orthogonal to $\ker C$. So $X = \alpha C$, $\alpha \neq 0$ (since we can choose B with $X \neq 0$ and no terms $e_1 \otimes e^2$, $e_2 \otimes e^1$).

Since e_- is an eigenvector of Y and Y is symmetric and traceless $Y = \begin{pmatrix} s & b_1 & -b_3 \\ 0 & s & 0 \\ 0 & b_3 & -2s \end{pmatrix}$.

From (8) $s = 0$ and from (9) $b_3 = -\alpha$ and since $\alpha \neq 0$ we can use (14) to put $b_1 = 0$.

In this way we obtain $B = X + Y + t\mathrm{id}_V = \begin{pmatrix} t & 0 & 2\alpha \\ 0 & t & 0 \\ 0 & 0 & t \end{pmatrix}$ and $b = \sqrt{2}\alpha(e_1 - e_2) \wedge k_3 + t(e_1 \wedge k_1 + e_2 \wedge k_2 + e_3 \wedge k_3)$. Using a one-parameter group of automorphisms of $\mathfrak{g}$

generated by vector e_3 one can transform this solution to solution with $\alpha = 0$. So we have $X = Y = 0$.

From equations (10)–(12): $\frac{2}{3}\operatorname{Tr}(AC) = \mu - 2t^2$, $AC = CA$, $AC + CA = \frac{2}{3}\operatorname{Tr}(AC)\mathrm{id}_V$. Since AC is not invertible $\mu = 2t^2$ and $\operatorname{Tr}(AC) = 0$. It follows that $A = 0$.

Using dilations: $(v, X) \mapsto (\lambda v, X)$, $\lambda \in R \setminus \{0\}$ we can assume that $t = 0$ or $t = 1$. This is solution (I) in our list.

5.2. *Solutions for* $C = 0$

In this case equations (6)–(13) reduce to the following:

$$\operatorname{Tr}(X^2) - \tfrac{1}{3}\operatorname{Tr}(Y^2) + 2t^2 = \mu \tag{15}$$

$$X(Y+t) + (Y+t)X = 0 \tag{16}$$

$$-4(Y^2 - \tfrac{1}{3}\operatorname{Tr}(Y^2)\mathrm{id}_V) - 2(YX - XY) + 4tY = 0 \tag{17}$$

$$\operatorname{Tr}(XA) = p. \tag{18}$$

We see that A is any antisymmetric matrix.

5.2.1. *Solutions for* $X \neq 0$.

Let us write equation (17) in the following form:

$$(X - Y + 2t)(Y + t) = 2t^2 - \frac{1}{3}\operatorname{Tr}(Y^2).$$

Since $X \neq 0$ and X is antisymmetric it follows that $2t^2 - \frac{1}{3}\operatorname{Tr}(Y^2) = 0$ (otherwise multiplying by $(Y+t)^{-1}$ we obtain $X = 0$). So for $X \neq 0$ we have the following equations:

$$\operatorname{Tr}(X^2) = \mu \tag{19}$$

$$X(Y+t) + (Y+t)X = 0 \tag{20}$$

$$(X - Y + 2t)(Y + t) = 0 \tag{21}$$

$$\operatorname{Tr}(Y^2) = 6t^2 \tag{22}$$

$$\operatorname{Tr}(XA) = p. \tag{23}$$

• $\dim\ker(Y+t) = 3$. Then $Y = 0$, $t = 0$ and X is any antisymmetric matrix. These are solutions (IIa–c) for $\rho = 0$, $\alpha \neq 0$. Using dilations one can put $\alpha = 1$.

• $\dim\ker(Y+t) = 2$. If $\eta|_{\ker(Y+t)}$ is nondegenerate then $Y + t$ has non-null eigenvector $v \in (\ker(Y+t))^{\perp}$ with eigenvalue $\lambda \neq 0$. Since $\ker(Y+t)$ is X-invariant we have $Xv = 0$. From (21) one has $\lambda = 3t \neq 0$. So for $\eta(v, v) > 0$ we can choose orthonormal basis (e_1, e_2, e_3) such that:

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\alpha \\ 0 & \alpha & 0 \end{pmatrix}, Y + t = \begin{pmatrix} 3t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

then $b = 3te_1 \wedge k_1 + \alpha(e_3 \wedge k_2 - e_2 \wedge k_3)$.

After rescaling this is solution (IIb) for $\alpha = 1$, $\rho \neq 0$.

For $\eta(v, v) < 0$: $X = \begin{pmatrix} 0 & \alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Y + t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3t \end{pmatrix}$.

This is solution (IIa) for $\alpha = 1$, $\rho \neq 0$.

If $\eta|_{\ker(Y+t)}$ is degenerate one can choose basis (e_-, e_+, e_3) with $e_-, e_3 \in \ker(Y+t)$.

Now $Y+t=\begin{pmatrix}0&\beta&0\\0&0&0\\0&0&0\end{pmatrix}$, $\beta\in R\setminus\{0\}$.

Because Y is traceless $t=0$ and from (21) $Xe_-=0$ and we obtain the family of solutions:

$$X=\begin{pmatrix}0&0&1\\0&0&0\\0&1&0\end{pmatrix}\qquad Y=\begin{pmatrix}0&\beta&0\\0&0&0\\0&0&0\end{pmatrix}\beta\in R\setminus\{0\}.$$

So $b=\frac{1}{\sqrt{2}}(e_3\wedge(k_1+k_2)+(e_1-e_2)\wedge k_3)+\frac{\beta}{2}(e_1-e_2)\wedge(k_1+k_2)$. This is solution (IIc) for $\alpha=1$, $\rho\neq 0$.

We can put $\rho>0$, since automorphisms of $\mathfrak{g}$ which on V are given by $P_i(v):=v-\eta_{ii}\eta(e_i,v)e_i$, where i = 1 for (IIa), i = 2 for (IIb) and i = 3 for (IIc) transform solutions (α,ρ) to $(\alpha,-\rho)$.

• $\dim\ker(Y+t)=1$. In this case $\ker(Y+t)$ has to be null subspace. Otherwise, since $\ker(Y+t)$ is X invariant, we obtain $Xv=0$ for $v\in\ker(Y+t)$. Now $Y+t$ is invertible on $v^\perp$ and this subspace is X-invariant. So we obtain $X=0$ from (21). As above, let us choose basis (e_-,e_+,e_3), $e_-\in\ker(Y+t)$. Then $Y+t=\begin{pmatrix}0&b_1&-b_3\\0&0&0\\0&b_3&3t\end{pmatrix}$.

(a) $e_-\in\ker X$.

So $X=\begin{pmatrix}0&0&1\\0&0&0\\0&1&0\end{pmatrix}$. From (20) $t=0$ and since $\ker(Y+t)$ is one-dimensional $b_3\neq 0$, so we can use (14) to put $b_1=0$. Now from (21) $b_3=-1$. This gives us a solution:

$B=X+Y+t\,\mathrm{id}_V=\begin{pmatrix}0&0&2\\0&0&0\\0&0&0\end{pmatrix}$ so $b=\sqrt{2}(e_1-e_2)\wedge k_3$. This is solution (IIIa).

(b) $Xe_-=\lambda e_-$, $\lambda\neq 0$

So $X=\begin{pmatrix}\lambda&0&0\\0&-\lambda&0\\0&0&0\end{pmatrix}$. Using (20): $b_3=0$, so $b_1,t\neq 0$. From (21): $\lambda=-3t\neq 0$. This gives family of solutions:

$$B=X+Y+t\,\mathrm{id}_V=\begin{pmatrix}-3t&b_1&0\\0&3t&0\\0&0&3t\end{pmatrix}\qquad \mu=18t^2,\ t,b_1\in R\setminus\{0\}.$$

So $b=\frac{b_1}{2}e_1\wedge k_1+(3t-\frac{b_1}{2})e_2\wedge k_1+(3t+\frac{b_1}{2})e_1\wedge k_2-\frac{b_1}{2}e_2\wedge k_2+3te_3\wedge k_3$. Dividing this by $\frac{b_1}{2}$ we obtain solutions (IIIb).

If $\rho>0(tb_1>0)$ we can use (14) to transform this solution to the following form:

$$B=\begin{pmatrix}-3t&0&s\\0&3t&0\\0&0&3t\end{pmatrix}$$

then $b=3t(e_1\wedge k_2+e_2\wedge k_1+e_3\wedge k_3)+\frac{s}{\sqrt{2}}(e_1-e_2)\wedge k_3=b_z+z\wedge Z+v\wedge Z$ for $z:=3te_3$, $Z:=k_3$, $v:=se_-$. This is the solution given in remark 4.

5.2.2. Solutions for $X=0$. From (15) $\mathrm{Tr}(Y^2)=6t^2-3\mu$. Substituting this to (17) we obtain the only equation for Y and t: $Y^2-tY+(\mu-2t^2)\mathrm{id}_V=0$

• $Y=0$, $t\in R$, $\mu=2t^2$. For $t=0$ we obtain solution five and for $t\neq 0$ (after rescaling) $b=e_1\wedge k_1+e_2\wedge k_2+e_3\wedge k_3$. It is easy to see, that for every $a\in\bigwedge^2 V$ there exist $v \in V$ such that: $v(b) = b + a$. So we can always put in the solution $a = 0$. This is solution (IV).

• Suppose $Y \neq 0$.

(a) $Yz = \lambda z$ for some positive z. So Y can be put into diagonal form and it is easy to see that there exists orthonormal basis (e_1, e_2, e_3) such that $Y + t$ is of one of the following forms:

$$Y + t = \begin{pmatrix} 3t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ or } Y + t = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3t \end{pmatrix} \text{ for } t \in R \setminus \{0\}.$$

In both cases $\mu = 0$. These are solutions (IIa) and b for $\alpha = 0$, $\rho \neq 0$.

(b) $Yv = \lambda v$ for some null vector v. Let us choose basis (e_-, e_+, e_3) such that $e_- = v$.

If $\lambda \neq 0$ then the basis can be chosen such that $Y = \begin{pmatrix} \lambda & b_1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & -2\lambda \end{pmatrix}$. Using the equation on Y one can see that $b_1 = 0$ and $\lambda = -t$ so this gives us no new solution.

If $\lambda = 0$ than it follows that $b_3 = 0$ and both t and μ are equal to 0. So we obtain another solution:$Y = \begin{pmatrix} 0 & b_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $b_1 \in R \setminus \{0\}$; $b = \frac{b_1}{2}(e_1 - e_2) \wedge (k_1 + k_2)$. This is solution (IIc) for $\alpha = 0$, $\rho \neq 0$.

6. Solutions for $E(3)$

From (6) it follows that $C = 0$.

6.1. Solutions for $X \neq 0$

• $\dim \ker(Y + t) = 3$. So $Y = 0, t = 0$ and X is any antisymmetric matrix. This is solution (I) for $\rho = 0$, $\alpha \neq 0$.

• $\dim \ker(Y + t) = 2$. It follows that $(\ker(Y + t))^\perp = \ker X$. In this way one obtains solutions (I) for α, $\rho \neq 0$.

• $\dim \ker(Y+t) = 1$. So $\ker(Y+t) = \ker X$ and since $Y+t$ is invertible on $(\ker(Y+t))^\perp$, $X = 0$ contrary to our assumption, so there is no solution of this type.

6.2. Solutions for $X = 0$

In this case, as in $P(3)$ we obtain only one equation for t and Y: $Y^2 - tY + (\mu - 2t^2)\mathrm{id}_V = 0$. It can easily be solved and one obtains solutions (I) for $\alpha = 0$, solution (II) and (III).

7. Appendix

We use the following notation: we denote elements of $\bigwedge^2 \mathfrak{g}$ by small letters and corresponding elements in $\mathrm{End}(V)$ by capital ones.

• $[c, c] \in \bigwedge^3 \mathfrak{h} \simeq W_0$ $\quad [\,,\,] : W_1 \otimes W_1 \longrightarrow W_0$, so it is proportional to F_0. Let us choose: $c_1 = k_1 \wedge k_2$, then $C_1 = \eta_{11}\eta_{22}k_3$. Computing $[c_1, c_1]$ one has $[C, C] = -\eta_{11}\eta_{22}\eta_{33}F_0(C \otimes C) = \eta_{11}\eta_{22}\eta_{33}\,\mathrm{Tr}(C^2)\mathrm{id}_V$.

• $[a, c] \in \bigwedge^2 V \otimes \mathfrak{h} \simeq W_0 \oplus W_1 \oplus W_2$ $\quad [\,,\,] : W_1 \otimes W_1 \longrightarrow W_0 \oplus W_1 \oplus W_2$, so $[A, C] = \alpha F_0(A \otimes C) + \beta F_1(A \otimes C) + \gamma F_2(A \otimes C)$. Let us choose $a_1 = e_1 \wedge e_2$, c_1—as above and $c_2 = k_1 \wedge k_3$. Then $C_2 = -\eta_{11}\eta_{33}k_2$, $A_1 = k_3$. We compute: $[a_1, c_1] = \eta_{22}e_1 \wedge e_3 \wedge k_2 - \eta_{11}e_2 \wedge e_3 \wedge k_1$ and $[a_1, c_2] = \eta_{22}e_1 \wedge e_3 \wedge k_3$, and find that $\alpha = -\frac{1}{3}$, $\beta = \gamma = -\frac{1}{2}$. In this way: $[A, C] = \frac{1}{3}\,\mathrm{Tr}(AC)\mathrm{id}_V + \frac{1}{2}(AC - CA) + \frac{1}{2}(AC + CA - \frac{2}{3}\,\mathrm{Tr}(AC)\mathrm{id}_V)$.

• $[b, c] \in V \otimes \bigwedge^2 \mathfrak{h} \simeq W_0 \oplus W_1 \oplus W_2$. Let $b = t + x + y$ be a decomposition (1). Then $[b, c] = [t, c] + [x, c] + [y, c]$.

(∗) $[t, c]$ $[\,,\,] : W_0 \otimes W_1 \longrightarrow W_0 \oplus W_1 \oplus W_2$, so it is proportional to F_1 and $F_1(T \otimes C) = 2TC$. Let us choose c_1—as above, $t_1 = e_1 \wedge k_1 + e_2 \wedge k_2 + e_3 \wedge k_3$, then $T_1 = \eta_{11}\eta_{22}\eta_{33}\mathrm{id}_V$. Computing $[t_1, c_1]$ we obtain 0, so $[T, C] = 0$.

(∗) $[x, c]$ $[\,,\,] : W_1 \otimes W_1 \longrightarrow W_0 \oplus W_1 \oplus W_2$. So $[X, C] = \alpha F_0(X \otimes C) + \beta F_1(X \otimes C) + \gamma F_2(X \otimes C)$. Let us choose: c_1, c_2—as above, $x_1 = -\eta_{11}e_2 \wedge k_1 + \eta_{22}e_\wedge k_2$, then $X_1 = \eta_{11}\eta_{22}\eta_{33}k_3$. Computing:

$[x_1, c_1] = -\eta_{22}\eta_{33}e_1 \wedge k_2 \wedge k_3 + \eta_{11}\eta_{33}e_2 \wedge k_1 \wedge k_3 - 2\eta_{11}\eta_{22}e_3 \wedge k_1 \wedge k_2$ and $[x_1, c_2] = -\eta_{11}\eta_{22}e_3 \wedge k_1 \wedge k_3$. It follows that $\alpha = -\frac{2}{3}$, $\beta = -\frac{1}{2}$, $\gamma = \frac{1}{2}$.

(∗) $[y, c]$ $[\,,\,] : W_1 \otimes W_2 \longrightarrow W_0 \oplus W_1 \oplus W_2$. Since the multiplicities of W_0, W_1, W_2 in $W_1 \otimes W_2$ are respectively 0,1 and 1, $[Y, C]$ is a linear combination of F_1 and F_2.

$[Y, C] = \beta F_1(Y \otimes C) + \gamma F_2(Y \otimes C)$. Choosing c_1—as above, $y_1 = \eta_{22}e_3 \wedge k_2 + \eta_{33}e_2 \wedge k_3$ we have $Y_1 = \eta_{11}\eta_{22}\eta_{33}(\eta_{22}e_3 \otimes e^2 + \eta_{33}e_2 \otimes e^3)$. Now we compute $[y_1, c_1] = \eta_{22}\eta_{33}(e_1 \wedge k_1 \wedge k_2 - 2e_3 \wedge k_2 \wedge k_3)$ It follows that $\beta = \frac{3}{2}$, $\gamma = \frac{1}{2}$.

So $[B, C] = \frac{2}{3}\,\mathrm{Tr}(XC)\mathrm{id}_V + \frac{1}{2}(3(YC + CY) - XC + CX) + \frac{1}{2}(-XC - CX + \frac{2}{3}\,\mathrm{Tr}(XC)\mathrm{id}_V + YC - CY)$.

• $[b, b] \in V \otimes \bigwedge^2 \mathfrak{h} \simeq W_0 \oplus W_1 \oplus W_2$.

$[b, b] = [x, x] + 2[x, y] + 2[x, t] + 2[y, t] + [y, y] + [t, t]$.

(∗) $[x, x]$ $[\,,\,] : W_1 \otimes W_1 \longrightarrow W_0 \oplus W_1 \oplus W_2$ is a symmetric intertwiner.

So $[X, X] = \alpha F_0(X \otimes X) + \gamma F_2(X \otimes X)$. Let x_1 be as above, then $[x_1, x_1] = 2\eta_{11}\eta_{22}(-\eta_{33}e_1 \wedge e_2 \wedge k_3 + \eta_{22}e_1 \wedge e_3 \wedge k_2 - \eta_{11}e_2 \wedge e_3 \wedge k_1)$. It follows that $\alpha = -1$, $\gamma = 0$.

(∗) $[y, y]$ $[\,,\,] : W_2 \otimes W_2 \longrightarrow W_0 \oplus W_1 \oplus W_2$. The multiplicities of W_0, W_1, W_2 in $W_2 \otimes W_2$ are equal to 1. Since the bracket is symmetric: $[Y, Y] = \alpha F_0(Y \otimes Y) + \gamma F_2(Y \otimes Y)$. Let y_1 be as above, then $[y_1, y_1] = 2\eta_{11}\eta_{33}(-\eta_{33}e_1 \wedge e_2 \wedge k_3 + \eta_{22}e_1 \wedge e_3 \wedge k_2 + \eta_{11}e_2 \wedge e_3 \wedge k_1)$. So $\alpha = -\frac{1}{3}$, $\gamma = -2$.

(∗) $[x, y]$ $[\,,\,] : W_1 \otimes W_2 \longrightarrow W_0 \oplus W_1 \oplus W_2$. Since the multiplicities of W_0, W_1, W_2 in $W_1 \otimes W_2$ are respectively 0,1, 1, $[X, Y]$ is a linear combination of F_1 and F_2: $[X, Y] = \beta F_1(X \otimes Y) + \gamma F_2(X \otimes Y)$. Now we have $[x_1, y_1] = 2\eta_{11}\eta_{22}\eta_{33}e_2 \wedge e_3 \wedge k_3$. It follows that $\beta = \gamma = -1$.

(∗) $[t, x]$ $[\,,\,] : W_0 \otimes W_1 \longrightarrow W_0 \oplus W_1 \oplus W_2$, so it is proportional to F_1 and $F_1(T \otimes X) = 2TX$. Computing: $[t_1, x_1] = 2\eta_{11}\eta_{22}(e_1 \wedge e_3 \wedge k_1 + e_2 \wedge e_3 \wedge k_2)$ we obtain $[X, T] = 2TX$.

(∗) $[t, y]$ $[\,,\,] : W_0 \otimes W_2 \longrightarrow W_0 \oplus W_1 \oplus W_2$, so it is proportional to F_2 and $F_2(T \otimes Y) = 2TY$. Computing: $[t_1, y_1] = 2\eta_{22}\eta_{33}(e_1 \wedge e_2 \wedge k_2 - e_1 \wedge e_3 \wedge k_3)$. So $[T, Y] = 2TY$.

(∗) $[t, t]$ $[\,,\,] : W_0 \otimes W_0 \longrightarrow W_0$, so it is proportional to F_0. Computing:

$[t_1, t_1] = 2(\eta_{33}e_1 \wedge e_2 \wedge k_3 - \eta_{22}e_1 \wedge e_3 \wedge k_2 + \eta_{11}e_2 \wedge e_3 \wedge k_1)$ and we obtain $[T, T] = \frac{2}{3}\,\mathrm{Tr}(T^2)\mathrm{id}_V$.

In this way: $[B, B] = (\mathrm{Tr}(X^2) - \frac{1}{3}\,\mathrm{Tr}(Y^2) + \frac{2}{3}\,\mathrm{Tr}(T^2))\mathrm{id}_V + 2(XY + YX + XT + TX) + -2(2Y^2 - \frac{2}{3}\,\mathrm{Tr}(Y^2)\mathrm{id}_V - XY + YX - 2TY)$.

• $[a, b] \in \bigwedge^3 V \simeq W_0$.

(∗) $[a, y]$ $[\,,\,] : W_1 \otimes W_2 \longrightarrow W_0$. So it is equal to 0.

(∗) $[a, t]$ $[\,,\,] : W_1 \otimes W_0 \longrightarrow W_0$. So it is equal to 0.

(∗) $[a, x]$ $[\,,\,] : W_1 \otimes W_1 \longrightarrow W_0$. So it is proportional to F_0. We compute:

$[a_1, x_1] = -2\eta_{11}\eta_{22}e_1 \wedge e_2 \wedge e_3$ and obtain $[A, X] = \eta_{11}\eta_{22}\eta_{33}\,\mathrm{Tr}(AX)\mathrm{id}_V$.
So $[A, B] = \eta_{11}\eta_{22}\eta_{33}\,\mathrm{Tr}(AB)\mathrm{id}_V = \eta_{11}\eta_{22}\eta_{33}\,\mathrm{Tr}(AX)\mathrm{id}_V$.

Putting all of the results together we obtain the following system of equations on $\mathrm{End}(V)$ equivalent to equation (4) and (5).

$$\mathrm{Tr}(C^2) = 0 \tag{24}$$

$$\mathrm{Tr}(XC) = 0 \tag{25}$$

$$XC - CX + 3(YC + CY) = 0 \tag{26}$$

$$(-XC - CX) + (YC - CY) = 0 \tag{27}$$

$$\tfrac{2}{3}\,\mathrm{Tr}(CA) + \mathrm{Tr}(X^2) - \tfrac{1}{3}\,\mathrm{Tr}(Y^2) + 2t^2 = \mu \tag{28}$$

$$-(CA - AC) + 2(XY + YX) + 4tX = 0 \tag{29}$$

$$CA + AC - \tfrac{2}{3}\,\mathrm{Tr}(CA)\mathrm{id}_V - 4(Y^2 - \tfrac{1}{3}\,\mathrm{Tr}(Y^2)\mathrm{id}_V) - 2(YX - XY) + 4tY = 0 \tag{30}$$

$$\mathrm{Tr}(XA) = p. \tag{31}$$

$$A^t = -A,\ C^t = -C,\ X^t = -X,\ Y^t = Y,\ \mathrm{Tr}(Y) = 0,\ T =: t\mathrm{id}_V.$$

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